A FORMULA FOR THE CHERN CLASSES OF SYMPLECTIC BLOW-UPS

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ABSTRACT. It is shown that the formula for the Chern classes (in the Chow ring) of blow-ups of algebraic varieties, due to Porteous and Lascu-Scott, also holds (in the cohomology ring) for blow-ups of symplectic and complex manifolds. This was used by the second-named author in her solution of the geography problem for 8-dimensional symplectic manifolds. The proof equally applies to real blow-ups of arbitrary manifolds and yields the corresponding blow-up formula for the Stiefel-Whitney classes. In the course of the argument the topological analogue of Grothendieck's *formule clef* in intersection theory is proved.

1. INTRODUCTION

Let (M, ω) be a symplectic manifold, J a tame almost complex structure, that is, $\omega(X, JX) > 0$ for any nonvanishing tangent vector X. The Chern classes of (M, ω) are defined as the Chern classes of the complex vector bundle (TM, J). Since the space of tame almost complex structures for a given symplectic form is non-empty and contractible (thus in particular connected), cf. [11, p. 65], this is a reasonable definition.

Given a symplectic submanifold N of M, the normal bundle of N in M carries a complex structure, and one can then define the blow-up $\widetilde{M}$ of M along N in analogy with the blow-up of complex manifolds along complex submanifolds.

The manifold $\widetilde{M}$ admits a symplectic form $\widetilde{\omega}$, which coincides with the pullback of ω outside a small neighbourhood of the exceptional divisor of the blow-up. The construction of $\widetilde{\omega}$, outlined in [5] and carried out in [10], depends on a number of choices, which may lead to non-isomorphic structures. However, the underlying tame almost complex structures are all homotopic. This allows us to speak unambiguously of the Chern classes of the symplectic blow-up.

In the algebraic setting, the Chern classes of the blown-up variety are given by a "blow-up formula" found by Porteous [13]. An alternative proof is due to Lascu and Scott [8], cf. also [9] and [7]. Here the Chern classes are understood as elements in the Chow ring (or intersection ring) of an algebraic variety, see [3] for a brief introduction. One naturally expects that the blow-up formula should carry over to the smooth topological setting, since many formulae in the Chow ring have analogues in the singular cohomology of manifolds. All the published proofs, however, depend to some degree on methods from algebraic geometry that lack an obvious topological correlate.
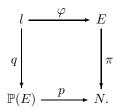
In the present paper we provide the necessary translation to the cohomology of smooth manifolds and use it to show that the blow-up formula (see Theorem 9) equally applies to the blow-up of symplectic and complex manifolds. Our proof of the blow-up formula is closest in spirit to the one in [9], but apart from references to the standard texts [1] and [2] it is completely self-contained. We also indicate how the proof carries over to real blow-ups, where one obtains the corresponding formula for the Stiefel-Whitney classes.

The proof in [9] relies in an essential way on Grothendieck's *formule clef* in intersection theory as proved in [7]. This is the part where the translation to the topological setting is least straightforward. Our proof of the topological analogue of the *formule clef* uses some ideas from Quillen's work [14] on complex cobordism theory.

2. The symplectic blow-up

We briefly recall the definition of the symplectic blow-up; for details see [10]. Consider a symplectic embedding $i\colon (N,\sigma)\to(M,\omega)$. We usually identify N with $i(N)\subset M$. The normal bundle E of N in M may be identified with the symplectic orthogonal bundle of $TN\subset TM$. Thus E carries a canonical symplectic bundle structure, given by the restriction of ω to each fibre, and hence a homotopically unique tame complex structure as well. With respect to this structure we consider the projectivisation $\mathbb{P}(E)$. Choose a tubular neighbourhood W of N in M. There exists a closed 2-form ρ on E which restricts to σ along the zero section and to the canonical symplectic form on each fibre, and with respect to which W may be symplectically identified with a neighbourhood V of the zero section of E. Let l be the tautological line bundle over $\mathbb{P}(E)$. Denote by q the bundle projecton $l\to\mathbb{P}(E)$ and by φ the projection $l\to E$, so that we have the commutative diagram

$$
\begin{array}{ccc}
l & \xrightarrow{\ \varphi\ } & E \\[2pt]
{\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\[2pt]
\mathbb{P}(E) & \xrightarrow{\ p\ } & N.
\end{array}
$$

Since φ is an isomorphism outside the zero section of l, one can make the following definition, cf. [10].

Definition. Set $\widetilde{V} := \varphi^{-1}(V)$; this is a disc sub-bundle (with fibres real 2-discs) of the complex line bundle l. The *blow-up* $\widetilde{M}$ of M along N is the manifold

$$\widetilde{M} := \overline{M-W}\cup_{\partial\widetilde{V}}\widetilde{V}.$$

We shall regard $\mathbb{P}(E)$ as the zero section of the disc bundle $\widetilde{V}\subset l$, and N as the zero section of $V\subset E$. The map φ gives us an identification of $\widetilde{V}-\mathbb{P}(E)$ with $V-N$. Thus, we may alternatively form $\widetilde{M}$ by identifying $M-N$ and $\widetilde{V}$ along $W-N\cong V-N\cong\widetilde{V}-\mathbb{P}(E)$. Either way, we see that there is a natural inclusion $\widetilde{\imath}\colon \mathbb{P}(E)\to\widetilde{M}$. This projective space $\mathbb{P}(E)$ is called the *exceptional divisor* of the blow-up.

Here is how to construct a symplectic form $\widetilde{\omega}$ on $\widetilde{M}$. On $M - W$ we set $\widetilde{\omega} = \omega$; one is then left with defining a symplectic form on $\widetilde{V}$ which equals $\varphi^*\rho$ near $\partial\widetilde{V}$. To do so, one considers a closed 2-form α on $\mathbb{P}(E)$ that restricts to the canonical symplectic form on each fibre of p, and that pulls back under q^* to a form on l that is exact outside the zero section $\mathbb{P}(E) \subset l$ (such a form may be obtained by the method of Thurston, cf. [11, Section 6.1]). Since $q^*\alpha$ is exact away from the zero section of l, one finds a 1-form β such that $q^*\alpha = d\beta$ on $l - \mathbb{P}(E)$. There is an $\overline{\varepsilon} > 0$, depending on ρ and α, such that for $\varepsilon \in (0, \overline{\varepsilon}]$ and with λ a radial bump function on $\widetilde{V}$ which equals 0 near the boundary, the form

$$\tilde{\rho} = \begin{cases} \varphi^*\rho + \varepsilon q^*\alpha & \text{on} \quad T\widetilde{M}|_{\mathbb{P}(E)}, \\ \varphi^*\rho + \varepsilon d(\lambda\beta) & \text{on} \quad \widetilde{V} - \mathbb{P}(E), \end{cases}$$

is nondegenerate on $\widetilde{V}$, and the form

$$\widetilde{\omega} = \begin{cases} \omega & \text{on} \quad M - W, \\ \tilde{\rho} & \text{on} \quad \widetilde{V}. \end{cases}$$

is a symplectic form on $\widetilde{M}$.

3. The blow-up diagram

With the symplectic identification of $V \subset E$ with $W \subset M$ understood, we can define a map $f\colon \widetilde{M} \to M$ by

$$f = \begin{cases} \text{id} & \text{on} \quad M - W, \\ \varphi & \text{on} \quad \widetilde{V}. \end{cases}$$

This map is a diffeomorphism outside $\mathbb{P}(E)$, and $\mathbb{P}(E) = f^{-1}(N)$. In particular, we have the commutative "blow-up diagram"

$$
\begin{array}{ccc}
\mathbb{P}(E) & \xrightarrow{\;\widetilde{\imath}\;} & \widetilde{M} \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} \\
N & \xrightarrow{\;i\;} & M.
\end{array}
$$

Notice that the normal bundle of $\widetilde{\imath}(\mathbb{P}(E))$ in $\widetilde{M}$ is isomorphic to l. In other words, we have the following short exact sequence of vector bundles:

$$(1) \qquad\qquad 0 \longrightarrow T\mathbb{P}(E) \longrightarrow \widetilde{\imath}^*T\widetilde{M} \longrightarrow l \longrightarrow 0.$$

When there is no ground for confusion, we shall identify $\mathbb{P}(E)$ with $\widetilde{\imath}(\mathbb{P}(E)) \subset \widetilde{M}$.

4. The Chern classes of symplectic blow-ups

The construction of a symplectic form on the blow-up of a symplectic manifold (M, ω) involves several choices and yields forms which are not necessarily isomorphic. Still, we would like to show that the Chern classes of such blown-up manifolds are well defined. First we show that we may choose a tame almost complex structure on M that is suitably adapted to the blow-up along N.

Lemma 1. *Let (M, ω) be a symplectic manifold, N a symplectic submanifold of M. Given any tame almost complex structure J_0 on the complement of N in M, one can find a tame almost complex structure J_M on M which coincides with J_0 outside a tubular neighbourhood of N and which is adapted to N, in the sense that TN is J_M-invariant and the almost complex structure $J_M|_{TN}$ is tame with respect to $\omega|_{TN}$.*

Proof. Let E be the normal bundle of N in M and W a tubular neighbourhood of N, symplectomorphic to a tubular neighbourhood of the zero section of E. Then W admits an almost complex structure J_W adapted to N.

The space $\mathcal{J}$ of ω-tame almost complex structures on $W - N$ is contractible, i.e. the identity map on $\mathcal{J}$ is homotopic to the constant map sending any almost complex structure J to $J_0|_{W-N}$. Let $F \colon \mathcal{J} \times I \to \mathcal{J}$ be the corresponding homotopy. We may assume that, for some small $\varepsilon > 0$, we have $F(J, t) = J$ for $t \leq \varepsilon$ and $F(J, t) = J_0$ for $t \geq 1 - \varepsilon$.

If we let $0 \leq t \leq 1$ denote the radial coordinate in W, so that $p \in W$ may be written as (x, v) in some bundle chart, with $x \in N$ and $\|v\| = t$, we can define an almost complex structure J_M on M as follows:

$$\begin{cases} J_M(x, v) = F(J_W, t)(x, v) & \text{for} \quad (x, v) \in W \\ J_M = J_0 & \text{on} \quad M - W \end{cases}$$

Then $J_M|_{TN} = J_W|_{TN}$ is again a tame almost complex structure for $\omega|_{TN}$, and $J_M = J_0$ outside W. $\qquad\square$

By construction, cf. [10, Lemma 3.3], the inclusion $\widetilde{\imath} \colon \mathbb{P}(E) \to \widetilde{M}$ is symplectic. With respect to a tame almost complex structure on $(\widetilde{M}, \widetilde{\omega})$ adapted to this symplectic submanifold as in Lemma 1, the sequence (1) can be read as an exact sequence of complex vector bundles. Moreover, the almost complex structures on $\widetilde{M}$ and M can be chosen in such a way that f is a pseudoholomorphic map.

We now want to show that such a tame almost complex structure does not depend, up to homotopy, on the choices in the construction of a symplectic form $\widetilde{\omega}$ on the blow-up $\widetilde{M}$.

First of all, the definition of $\widetilde{M}$ does not depend on the choice of (tame) complex bundle structure on the normal bundle E of N in M. This follows from all such choices being homotopic and general bundle theory, cf. [6, Section 4.9].

Now, the construction of $\widetilde{\omega}$ involved the choice of 2-forms ρ and α, a bump function λ, and an $\varepsilon > 0$ in an allowable range $(0, \overline{\varepsilon}]$ depending on ρ and α. The conditions on ρ and α are convex. Thus, given two such choices $\rho_i, \alpha_i, \ i = 0, 1$ (and a corresponding β_i), as well as bump functions λ_i, one can define $\rho_t, \alpha_t, \beta_t, \lambda_t$, $t \in [0, 1]$, as the respective convex linear combinations $\rho_t = (1 - t)\rho_0 + t\rho_1$ etc. Since the nondegeneracy condition on $\tilde{\rho}_t$ is an open condition, and the parameter space $[0, 1]$ is compact, we can find an $\varepsilon > 0$ (and smaller than $\overline{\varepsilon}_0, \overline{\varepsilon}_1$) such that $\tilde{\rho}_t$ is symplectic for all $t \in [0, 1]$. Moreover, varying ε in the allowable range $(0, \overline{\varepsilon}]$ gives likewise a family of symplectic forms. Thus, the corresponding symplectic forms $\widetilde{\omega}_0$ and $\widetilde{\omega}_1$ on $\widetilde{M}$ are homotopic through (noncohomologous) symplectic forms, and therefore induce homotopic tame almost complex structures.

Notice that we could choose an $\varepsilon > 0$ only because of our restriction to a compact family (the convex linear interpolation between two choices). It is not clear that there is an $\varepsilon > 0$ such that one can interpolate between all $\widetilde{\omega}$ through forms corresponding to the same ε. If this were possible, the interpolation would be through cohomologous forms, and thus by Moser stability all 'ε-blow-ups' would be symplectomorphic. This is not known, in general, cf. [10, p. 250].

5. Some cohomological lemmas

We start by proving some general results, which apply in particular to the blow-up situation. Recall that for any map $f \colon N{\to}M$ of smooth, compact, oriented manifolds of dimension n and m, respectively, one can define a "shriek" or "transfer" homomorphism

$$f^! \colon H^{n-p}(N, \partial N) \longrightarrow H^{m-p}(M, \partial M)$$

by $f^! = PD_M \circ f_* \circ PD_N^{-1}$. Here PD denotes the Poincaré duality isomorphism from homology to cohomology. Likewise, one can define the shriek homomorphism on absolute cohomology groups, provided f takes the boundary ∂N into ∂M. There is an analogous shriek map $f_!$ on homology, but this will not be used in the present paper. We shall frequently apply the so-called *projection formula*

$$f^!(f^*(a) \cup b) = a \cup f^!(b)$$

for $a \in H^*(M)$ or $H^*(M, \partial M)$ and $b \in H^*(N, \partial N)$. For more details about this and the following statements see [2, Sections VI.11, 12 and 14].

Notation. We use the topologist's convention to label the shriek map on cohomology with a superscript. More algebraically minded people sometimes write it with a subscript, emphasising the covariance of this map. To make matters worse, the corresponding map on Chow rings is written as f_*, whereas the (upper and lower) shriek maps on Chow rings have a different meaning altogether, cf. [3].

If W is a k-disc bundle over a manifold N of dimension n, with projection $\pi \colon W{\to}N$, and with $i_0 \colon N{\to}W$ the inclusion of N in W as the zero section, the Thom class of W is defined by

$$\tau = i_0^!(1) = PD_W\, i_{0*}[N] \in H^k(W, \partial W).$$

The Thom isomorphism theorem states that $i_0^!$ is an isomorphism that coincides with the composition

$$i_0^! \colon H^p(N) \xrightarrow{\pi^*} H^p(W) \xrightarrow{\cup \tau} H^{p+k}(W, \partial W).$$

If, more generally, $i \colon N{\to}M$ is a smooth codimension k embedding of manifolds, possibly with boundaries (in which case N is required to meet ∂M transversely in ∂N), the Thom class of the inclusion is defined to be

$$\tau_N^M = i^!(1) = PD_M i_*[N] \in H^k(M).$$

Its pull-back under the inclusion i is the Euler class of the normal bundle: $\chi_N^M = i^*\tau_N^M \in H^k(N)$. If we denote by W a closed tubular neighbourhood of N in M and identify it with a k-disc sub-bundle of the normal bundle of N in M, the

Thom class τ_N^M is the image of the Thom class τ of W under the composition of homomorphisms

$$H^k(W, \partial W) \xrightarrow{\mathrm{exc}^{-1}} H^k(M, M - N) \longrightarrow H^k(M).$$

Here exc denotes the excision isomorphism induced by the inclusion of pairs

$$(W, \partial W) \rightarrow (M, M - N).$$

(On the level of (co-)homology, we do not need to distinguish between the pairs $(W, \partial W)$ and $(W, W - N)$.) The second homomorphism is induced by the inclusion of $(M, \emptyset)$ in $(M, M - N)$.

From now on, M and N will always be closed manifolds. We regard N as a submanifold of M and interpret the embedding $i\colon N \to M$ as an inclusion map. The name of the following lemma (and several other results below) derives from the corresponding statement in intersection theory.

Lemma 2 (Excision Lemma). *Let $i\colon N \to M$ be an inclusion of smooth closed manifolds, $i_c\colon M - N \to M$ the inclusion of the complement of N. Suppose $\lambda \in H^*(M)$ satisfies $i_c^*(\lambda) = 0$. Then there exists a class $\beta \in H^*(N)$ such that $i^!(\beta) = \lambda$.*

Proof. As before we write $i_0\colon N \to W \subset M$ for the inclusion of N in a tubular neighbourhood W. Write

$$j_M\colon (M, \emptyset) \longrightarrow (M, M - N)$$

for the inclusion of pairs. Consider the diagram

$$
\begin{array}{ccccc}
H^*(N) & \xrightarrow{\ i^!\ } & H^*(M) & \xrightarrow{\ i_c^*\ } & H^*(M - N) \\
& \searrow{\scriptstyle \mathrm{exc}^{-1} i_0^!} & \uparrow{\scriptstyle j_M^*} & & \\
& & H^*(M, M - N). & &
\end{array}
$$

Since the sequence

$$H^*(M, M - N) \xrightarrow{j_M^*} H^*(M) \xrightarrow{i_c^*} H^*(M - N)$$

is exact, and $\mathrm{exc}^{-1} i_0^!$ is an isomorphism, it suffices to show that the diagram above is commutative, i.e. $j_M^* \mathrm{exc}^{-1} i_0^! = i^!$. With the inclusion $i_1\colon W \to M$ we have $i = i_1 i_0$, hence $i^! = i_1^! i_0^!$. Since $i_0^!$ is an isomorphism, what we want to show is

$$(2) \qquad\qquad\qquad j_M^* \mathrm{exc}^{-1} = i_1^!,$$

in other words, that the following diagram is commutative:

$$
\begin{array}{ccccc}
H^*(W, \partial W) & \xrightarrow{\ \mathrm{exc}^{-1}\ } & H^*(M, M - N) & \xrightarrow{\ j_M^*\ } & H^*(M) \\
\downarrow{\scriptstyle PD_W^{-1}} & & & & \downarrow{\scriptstyle \equiv} \\
H_*(W) & \xrightarrow{\ i_{1*}\ } & H_*(M) & \xrightarrow{\ PD_M\ } & H^*(M).
\end{array}
$$

Since $H^*(W, \partial W) \cong H^*(W, W - N)$, a given cohomology class $w \in H^*(W, \partial W)$ can be represented by a cochain on W that vanishes on singular simplices contained in $W - N$. Hence we can write $w = \mathrm{exc}(\widetilde{w})$ with $\widetilde{w} \in H^*(M, M - N)$ represented by a cochain that vanishes on singular simplices contained in $M - N$. Notice that if we write, by slight abuse of notation, i_1 also for the inclusion of pairs

$$i_1 \colon (W, \partial W) \longrightarrow (M, M - N),$$

then $\mathrm{exc} = i_1^*$. In the following calculations we use i_1 in both senses.

By what we just said about $\widetilde{w}$, we have

$$\widetilde{w} \cap i_{1*}[W] = j_M^*(\widetilde{w}) \cap [M],$$

since we may represent the fundamental classes $[W] \in H_m(W, \partial W)$ and $[M] \in H_m(M)$ by singular chains that differ only by singular simplices contained in $M - W$. Hence

$$
\begin{aligned}
PD_M i_{1*} PD_W^{-1}(w) &= PD_M i_{1*}(w \cap [W]) \\
&= PD_M i_{1*}(i_1^*(\widetilde{w}) \cap [W]) \\
&= PD_M(\widetilde{w} \cap i_{1*}[W]) \\
&= PD_M(j_M^*(\widetilde{w}) \cap [M]) \\
&= j_M^*(\widetilde{w}) \\
&= j_M^* \mathrm{exc}^{-1}(w).
\end{aligned}
$$

This concludes the proof. $\qquad\square$

Remark. Equation (2) explains the statement about the relation between τ_N^M and τ made before the excision lemma:

$$j_M^* \mathrm{exc}^{-1}(\tau) = i_1^! i_0^!(1) = i^!(1) = \tau_N^M.$$

Up to this point, N was an arbitrary submanifold of M. From now on, we only consider the special set-up described in Section 2. We write $2r$ for the rank of the normal bundle E of N in M, and $c_r(E)$ for the top Chern class (or Euler class) of this bundle. Then

$$c_r(E) = \chi_N^M = i^* \tau_N^M = i^* i^!(1).$$

The following lemma generalises this formula.

Lemma 3 (Self-intersection formula)**.** *For any $y \in H^*(N)$ we have*

$$y \cup c_r(E) = i^* i^!(y).$$

Proof. Our notation is as in the proof of Lemma 2. Let j_W be the inclusion of pairs $(W, \emptyset) \to (W, W - N)$. Then we have the commutative diagram

$$
\begin{array}{ccc}
H^*(M, M - N) & \xrightarrow{\;j_M^*\;} & H^*(M) \\
\Big\downarrow{\scriptstyle \mathrm{exc} = i_1^*} & & \Big\downarrow{\scriptstyle i_1^*} \\
H^*(W, W - N) & \xrightarrow{\;j_W^*\;} & H^*(W).
\end{array}
$$

Denote by
$$\tau \in H^{2r}(W, \partial W) \equiv H^{2r}(W, W - N)$$
the Thom class $PD_W i_{0*}[N]$. Notice that $i_0^* j_W^*(\tau) = c_r(E) \in H^{2r}(N)$, cf. [2, p. 378].

For $w \in H^*(W, \partial W)$ we have, by equation (2),
$$i_1^* i_1^!(w) = i_1^* j_M^* \mathrm{exc}^{-1}(w) = j_W^*(w).$$
Given $y \in H^*(N)$, we can apply this identity to $w = \pi^*(y) \cup \tau$. We obtain
$$
\begin{aligned}
i^* i^!(y) &= i_0^* i_1^* i_1^! i_0^!(y) = i_0^* i_1^* i_1^!(\pi^*(y) \cup \tau) \\
&= i_0^* j_W^*(\pi^*(y) \cup \tau) = i_0^*(\pi^*(y) \cup j_W^*(\tau)) \\
&= y \cup i_0^* j_W^*(\tau) = y \cup c_r(E).
\end{aligned}
$$
This is the claimed formula.   □

In the following lemma (and throughout the remainder this paper) we write
$$\xi = -c_1(l) \in H^2(\mathbb{P}(E))$$
for the first Chern class of the dual tautological line bundle l^*; this is the sign convention of [9] and motivated by the fact that this ξ is the positive generator of $H^2(\mathbb{P}(E))$.

Lemma 4. *Suppose $\widetilde{y} \in H^*(\widetilde{M})$ has the property that $\widetilde{\imath}^*(\widetilde{y}) = -\bar{y} \cup \xi$ for some $\bar{y} \in H^*(\mathbb{P}(E))$. Then $\widetilde{y} = \widetilde{\imath}^!(\bar{y}) + \lambda$ with $\widetilde{\imath}^*(\lambda) = 0$.*

Proof. We can write $\widetilde{y} = \widetilde{\imath}^!(\bar{y}) + (\widetilde{y} - \widetilde{\imath}^!(\bar{y}))$. By the preceding lemma, applied to the normal bundle l of $\mathbb{P}(E) \subset \widetilde{M}$, with top Chern class $-\xi$, we have $\widetilde{\imath}^* \widetilde{\imath}^!(\bar{y}) = -\bar{y} \cup \xi$. Hence
$$\widetilde{\imath}^*(\widetilde{y} - \widetilde{\imath}^!(\bar{y})) = \widetilde{\imath}^*(\widetilde{y}) - \widetilde{\imath}^* \widetilde{\imath}^!(\bar{y}) = -\bar{y} \cup \xi + \bar{y} \cup \xi = 0. \quad \square$$

6. The *formule clef*

The line bundle l may be regarded as a sub-bundle of the pull-back bundle p^*E over $\mathbb{P}(E)$. The quotient bundle Q is defined by the short exact sequence
$$(3) \qquad\qquad\qquad 0 \longrightarrow l \longrightarrow p^*E \longrightarrow Q \longrightarrow 0.$$
Recall, cf. [1, eqn. (20.7)], that the cohomology ring $H^*(\mathbb{P}(E))$ can be described as
$$H^*(\mathbb{P}(E)) = H^*(N)[\xi]/(\xi^r + p^*c_1(E)\xi^{r-1} + \cdots + p^*c_r(E)),$$
where from now on we write the cup product as an ordinary product. The relation

$$(4) \qquad\qquad \xi^r + p^*c_1(E)\xi^{r-1} + \cdots + p^*c_r(E) = 0$$

in $H^*(\mathbb{P}(E))$ will be called the *fundamental relation*. From the exact sequence (3) we have, with c denoting the total Chern class,
$$p^*c(E) = c(Q)(1 - \xi).$$
By multiplying this equation by $(1 + \xi + \cdots + \xi^{r-1})$, using the fundamental relation, and collecting terms of degree $2r - 2$ we find
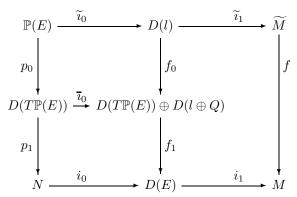$$(5) \qquad\qquad c_{r-1}(Q) = \xi^{r-1} + p^*c_1(E)\xi^{r-2} + \cdots + p^*c_{r-1}(E).$$

The following key formula, in the algebraic geometric setting originally conjectured by Grothendieck and proved in [7], gives an important tool for computing in the cohomology rings of the manifolds appearing in the blow-up diagram.

Proposition 5 (Grothendieck's *formule clef*). *For any class $y \in H^*(N)$ we have*

$$f^* i^!(y) = \widetilde{i}^!(p^*(y)c_{r-1}(Q)).$$

The proof of this proposition will take up the rest of this section. Consider the following commutative diagram, where we write $D(\mathcal{E})$ for the disc bundle associated with a vector bundle $\mathcal{E}$. The disc bundles $D(E), D(l)$ will be identified with tubular neighbourhoods of $N, \mathbb{P}(E)$ in $M, \widetilde{M}$, respectively. (In other words, $D(E) = W$ and $D(l) = \widetilde{V}$ in our previous notation.) The maps in this diagram that have not yet been defined will be explained presently.

$$
\begin{array}{ccccc}
\mathbb{P}(E) & \xrightarrow{\widetilde{\imath}_0} & D(l) & \xrightarrow{\widetilde{\imath}_1} & \widetilde{M} \\
\downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\
D(T\mathbb{P}(E)) & \xrightarrow{\overline{\imath}_0} & D(T\mathbb{P}(E)) \oplus D(l \oplus Q) & & \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f_1} & & \\
N & \xrightarrow{i_0} & D(E) & \xrightarrow{i_1} & M
\end{array}
$$

Here p_0 is the inclusion of $\mathbb{P}(E)$ as the zero section in its tangent disc bundle; p_1 is the natural projection of $D(T\mathbb{P}(E))$ onto $\mathbb{P}(E)$, followed by $p\colon \mathbb{P}(E) \to N$. This means that p factors as $p = p_1 \circ p_0$.

With $D(T\mathbb{P}(E)) \oplus D(l \oplus Q)$ we denote the bundle over $\mathbb{P}(E)$ whose fibre over a point $x \in \mathbb{P}(E)$ is the product of the unit disc in $T_x\mathbb{P}(E)$ with that in $(l \oplus Q)_x = (p^*E)_x = E_{p(x)}$. The maps f_0 and f_1 are the obvious ones. The restriction of f to $D(l)$ factorises as $f_1 \circ f_0$, so that the square on the right is indeed commutative.

The square on the top left is commutative since all the maps in that square are inclusion maps. To see the commutativity of the square on the bottom left, recall that

$$p^*E = \{(x,v) \in \mathbb{P}(E) \times E \colon p(x) = \pi(v)\},$$

where $\pi\colon E \to N$ denotes the bundle projection as before. Then for $x \in \mathbb{P}(E)$ and $t \in D(T_x\mathbb{P}(E))$ we have

$$i_0 p_1(t) = i_0 p(x) = 0 \in E_{p(x)},$$

and likewise

$$f_1 \overline{\imath}_0(t) = f_1(t, x, 0 \in E_{p(x)}) = 0 \in E_{p(x)}.$$

In the following lemma and its proof we write f not only for the map $\widetilde{M} \to M$, but also for its restriction to subspaces or pairs of subspaces.

Lemma 6. *For $w \in H^*(D(E), \partial D(E))$ we have*

$$f^* i_1^!(w) = \widetilde{i}_1^! f^*(w).$$

Proof. We have $i_1^! = j_M^* \mathrm{exc}^{-1}$, see equation (2) in the proof of the excision lemma. Likewise, with $\widetilde{j}_{\widetilde{M}}$ denoting the inclusion of pairs

$$\widetilde{j}_{\widetilde{M}} \colon (\widetilde{M}, \emptyset) \longrightarrow (\widetilde{M}, \widetilde{M} - \mathbb{P}(E)),$$

we have $\widetilde{i}_1^! = \widetilde{j}_{\widetilde{M}}^* \mathrm{exc}^{-1}$, where exc now stands for the excision isomorphism

$$H^*(\widetilde{M}, \widetilde{M} - \mathbb{P}(E)) \longrightarrow H^*(D(l), \partial D(l)).$$

These excision isomorphisms, being induced by inclusions, commute with f^*, and so do j_M^* and $\widetilde{j}_{\widetilde{M}}^*$. This proves the lemma. $\qquad\qquad\square$

Next we deal with the square on the bottom left. This is in fact a cartesian square, i.e. $D(T\mathbb{P}(E))$ may be regarded as the fibre product (or pull-back)

$$N \times_{D(E)} (D(T\mathbb{P}(E)) \oplus D(p^*E)) :=$$
$$\{(n; t, x, v) \in N \times D(T\mathbb{P}(E)) \times \mathbb{P}(E) \times D(E) \colon$$
$$t \in T_x\mathbb{P}(E),\, p(x) = \pi(v),\, i_0(n) = f_1(t, x, v)\}.$$

Indeed, we have $f_1(t, x, v) = v$, so the defining equation for the fibre product becomes $i_0(n) = v$, which implies $v = 0 \in E_n$. From $p(x) = \pi(v)$ we then get $n = p(x)$. So the isomorphism of the fibre product $N \times_{D(E)} (D(T\mathbb{P}(E)) \oplus D(p^*E))$ with the disc bundle $D(T\mathbb{P}(E))$ is given by

$$(n = p(x); t \in T_x\mathbb{P}(E), x, 0 \in E_n) \longmapsto t \in T_x\mathbb{P}(E),$$

which has an obvious inverse.

The crucial point for us, however, is the transversality of the maps i_0 and f_1. Recall that the Thom class of a disc bundle $\mathcal{D}$ over a closed manifold is the class in $H^k(\mathcal{D}, \partial\mathcal{D})$, with k denoting the fibre dimension, characterised by the fact that it restricts on each fibre $\mathcal{D}_x$ to the positive generator of $H^k(\mathcal{D}_x, \partial\mathcal{D}_x)$. (All our bundles are complex and thus carry natural fibre orientations.)

In our situation we are dealing with a disc bundle $\mathcal{D} := D(T\mathbb{P}(E)) \oplus D(p^*E)$ over the manifold $D(T\mathbb{P}(E))$, which itself has boundary. So $\mathcal{D}$ is a manifold with corners, but it is still possible to define a Thom class in this setting:

Write the boundary of $\mathcal{D}$ as $\partial\mathcal{D} = \partial_B \cup \partial_F$, where the subscripts B, F denote the part of the boundary corresponding to the boundary of the base and fibre, respectively. The intersection $\partial_B \cap \partial_F$ is the codimension 2 'corner' of $\mathcal{D}$. The cap product with the fundamental class of $\mathcal{D}$ gives a duality isomorphism PD^{-1} from $H^*(\mathcal{D}, \partial_F)$ to $H_*(\mathcal{D}, \partial_B)$, see [2, p. 358]. So we may define the Thom class of $\mathcal{D}$ as before, but now this is a cohomology class in $H^k(\mathcal{D}, \partial_F)$.

The statements about the Thom isomorphism and the characterisation of the Thom class remain valid with the obvious changes. Thus, if we write $\overline{\pi}$ for the bundle projection

$$\overline{\pi} \colon \mathcal{D} := D(T\mathbb{P}(E)) \oplus D(p^*E) \longrightarrow D(T\mathbb{P}(E)) =: B$$

and $\overline{\tau} \in H^{2r}(\mathcal{D}, \partial_F)$ for the Thom class of this disc bundle, the composition

$$H^p(B) \xrightarrow{\overline{\pi}^*} H^p(\mathcal{D}) \xrightarrow{\cup \overline{\tau}} H^{p+2r}(\mathcal{D}, \partial_F)$$

is an isomorphism that coincides with

$$\overline{i}_0^! \colon H^p(B) \xrightarrow{PD^{-1}} H_{b-p}(B, \partial B) \xrightarrow{i_{0*}} H_{b-p}(\mathcal{D}, \partial_B) \xrightarrow{PD} H^{p+2r}(\mathcal{D}, \partial_F),$$

where we wrote b for the dimension of the base.

The characterisation of the Thom class $\overline{\tau}$ as the class that restricts to the appropriate cohomology generator on each fibre likewise remains valid — argue as in the case where the base is a closed manifold. In our situation this implies $f_1^*(\tau) = \overline{\tau}$, with f_1 regarded as the map $(\mathcal{D}, \partial_F) \to (D(E), \partial D(E))$.

Lemma 7 (Pull-back)**.** *We have $f_1^* i_0^! = \overline{i}_0^! p_1^*$ as homomorphisms from $H^*(N)$ to $H^*(\mathcal{D}, \partial_F)$.*

Proof. From the definitions it is obvious that $\pi \circ f_1 = p_1 \circ \overline{\pi}$. Hence, for any class $y \in H^*(N)$ and with $f_1^*(\tau) = \overline{\tau}$ we get

$$f_1^* i_0^!(y) = f_1^*(\pi^*(y) \cup \tau) = \overline{\pi}^* p_1^*(y) \cup \overline{\tau} = \overline{i}_0^! p_1^*(y). \quad \square$$

Finally, we turn to the square on the top left. For the purposes of our cohomological computations we may replace the bundle $D(T\mathbb{P}(E)) \oplus D(l \oplus Q)$ by $D(T\mathbb{P}(E)) \oplus D(l) \oplus D(Q)$. Then the maps $\overline{i}_0$ and f_0 can be factorised as

$$\overline{i}_0 \colon D(T\mathbb{P}(E)) \xrightarrow{k_1} D(T\mathbb{P}(E)) \oplus D(l) \xrightarrow{k} D(T\mathbb{P}(E)) \oplus D(l) \oplus D(Q)$$

and

$$f_0 \colon D(l) \xrightarrow{k_2} D(T\mathbb{P}(E)) \oplus D(l) \xrightarrow{k} D(T\mathbb{P}(E)) \oplus D(l) \oplus D(Q).$$

The commutative diagram

$$
\begin{array}{ccc}
\mathbb{P}(E) & \xrightarrow{\ \widetilde{i}_0\ } & D(l) \\
\downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle k_2} \\
D(T\mathbb{P}(E)) & \xrightarrow{\ k_1\ } & D(T\mathbb{P}(E)) \oplus D(l)
\end{array}
$$

again constitutes a cartesian square, and the maps k_1 and k_2 are transverse to each other. By the analogue of the preceding lemma, we have

$$k_2^* k_1^! = \widetilde{i}_0^! p_0^*.$$

The following lemma and its proof are analogous to an argument employed by Quillen [14] in his study of the complex cobordism ring.

Lemma 8 (Clean intersection formula)**.** *For any class $\overline{y} \in H^*(D(T\mathbb{P}(E)))$ we have*

$$f_0^* \overline{i}_0^!(\overline{y}) = \widetilde{i}_0^!(p_0^*(\overline{y}) c_{r-1}(Q)).$$

Proof. Write $\widetilde{\pi}$ for the bundle projection $D(l) \to \mathbb{P}(E)$. Denote by ν_k the normal bundle of $D(T\mathbb{P}(E)) \oplus D(l)$ in $D(T\mathbb{P}(E)) \oplus D(l) \oplus D(Q)$. Then $k_2^* \nu_k = \widetilde{\pi}^* Q$.

We then compute

$$
\begin{aligned}
f_0^* \overline{\imath}_0^!(\overline{y}) &= k_2^* k^* k^! k_1^!(\overline{y}) \\
&= k_2^*(k_1^!(\overline{y}) c_{r-1}(\nu_k)) && \text{(self-intersection)} \\
&= k_2^* k_1^!(\overline{y}) k_2^* c_{r-1}(\nu_k) \\
&= k_2^* k_1^!(\overline{y}) \widetilde{\pi}^* c_{r-1}(Q) \\
&= \widetilde{\imath}_0^! p_0^*(\overline{y}) \widetilde{\pi}^* c_{r-1}(Q) && \text{(pull-back)} \\
&= \widetilde{\imath}_0^!(p_0^*(\overline{y}) \widetilde{\imath}_0^* \widetilde{\pi}^* c_{r-1}(Q)) && \text{(projection formula)} \\
&= \widetilde{\imath}_0^!(p_0^*(\overline{y}) c_{r-1}(Q)),
\end{aligned}
$$

which is the desired formula. $\qquad\square$

It is now a simple matter to prove the *formule clef.*

Proof of Proposition 5. We have

$$
\begin{aligned}
f^* i^!(y) &= f^* i_1^! i_0^!(y) \\
&= \widetilde{\imath}_1^! f^* i_0^!(y) && \text{(Lemma 6)} \\
&= \widetilde{\imath}_1^! f_0^* f_1^* i_0^!(y) \\
&= \widetilde{\imath}_1^! f_0^* \overline{\imath}_0^! p_1^*(y) && \text{(pull-back)} \\
&= \widetilde{\imath}_1^! \widetilde{\imath}_0^!(p_0^* p_1^*(y) c_{r-1}(Q)) && \text{(clean intersection)} \\
&= \widetilde{\imath}^!(p^*(y) c_{r-1}(Q)),
\end{aligned}
$$

as was to be shown. $\qquad\square$

7. The blow-up formula

Write $c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \dots$ for the total Chern class of a complex vector bundle $\mathcal{E}$. If $\mathcal{E}$ is the tangent bundle TB of some manifold B, we write $c(B)$ instead of $c(TB)$.

The expression

$$
\sum_{i=0}^{r} p^* c_i(E)(1+\xi)^{r-i}(1-\xi) - p^* c(E) \in H^*(\mathbb{P}(E))
$$

is obviously of strictly positive degree in ξ, so it makes sense to divide this term by ξ.

Theorem 9 (Blow-up formula). *Let N be a closed symplectic (resp. complex) submanifold of (real) codimension $2r$ in a symplectic (resp. complex) manifold M. Write E for the normal bundle of N in M with its natural complex bundle structure, and $\widetilde{M}$ for the symplectic (resp. complex) blow-up of M along N. Let ξ be the first Chern class of the dual tautological line bundle over the projectivised bundle $\mathbb{P}(E)$. With $f, \widetilde{i}, p$ the maps from the blow-up diagram in Section 2 we have*

$$
c(\widetilde{M}) - f^* c(M) = -\widetilde{\imath}^! \left[p^* c(N) \cdot \frac{1}{\xi} \left(\sum_{i=0}^{r} p^* c_i(E)(1+\xi)^{r-i}(1-\xi) - p^* c(E) \right) \right].
$$

The idea for proving this theorem is as in [9] and goes back to Mumford. We first prove a weak version of the blow-up formula, which differs from the ultimate version by a term of the form $f^*i^!(\beta)$ with some class $\beta \in H^*(N)$. By applying this weak blow-up formula to the blow-up of $M \times S^2$ along $N \subset M \subset M \times S^2$, one arrives at Theorem 9.

We start with a lemma that allows us to write certain cohomology classes in $H^*(\widetilde{M})$ in the form $f^*i^!(\beta)$. This is in fact the only place where we have to rely on the *formule clef* (except for an application in Section 8).

Lemma 10. *If $\gamma \in H^*(\mathbb{P}(E))$ is a cohomology class that satisfies $\gamma\xi = 0$ — which by the self-intersection formula is equivalent to saying $\widetilde{\imath}^*\widetilde{\imath}^!(\gamma) = 0$ —, then there is a class $\beta \in H^*(N)$ such that $\widetilde{\imath}^!(\gamma) = f^*i^!(\beta)$.*

Proof. By the structure of the cohomology ring $H^*(\mathbb{P}(E))$ described at the beginning of the preceding section, any class $\gamma \in H^*(\mathbb{P}(E))$ can be described uniquely in the form

$$\gamma = p^*(\beta_1)\xi^{r-1} + \cdots + p^*(\beta_{r-1})\xi + p^*(\beta_r)$$

with suitable $\beta_i \in H^*(N)$. Use the fundamental relation (4) in $H^*(\mathbb{P}(E))$ to rewrite the equation $\gamma\xi = 0$ as

$$p^*(\beta_2 - \beta_1 c_1(E))\xi^{r-1} + \cdots + p^*(\beta_r - \beta_1 c_{r-1}(E))\xi - p^*(\beta_1 c_r(E)) = 0.$$

Since p^* is injective, this implies

$$\beta_2 = \beta_1 c_1(E), \ldots, \beta_r = \beta_1 c_{r-1}(E).$$

With equation (5) this yields $p^*(\beta_1)c_{r-1}(Q) = \gamma$. By applying the *formule clef* we obtain $\widetilde{\imath}^!(\gamma) = f^*i^!(\beta_1)$, so $\beta := \beta_1$ is the desired class. $\qquad\square$

In order to prove the (weak) blow-up formula, we begin by computing the result of applying $\widetilde{\imath}^*$ to the left-hand side of the formula in Theorem 9. From the exact sequence of complex vector bundles

$$0 \longrightarrow TN \longrightarrow i^*TM \longrightarrow E \longrightarrow 0$$

we get

$$\widetilde{\imath}^* f^* c(M) = p^* i^* c(M) = p^* c(N) p^* c(E).$$

Likewise, from the exact sequence (1) we have

$$\widetilde{\imath}^* c(\widetilde{M}) = c(\mathbb{P}(E))c(l) = c(\mathbb{P}(E)) \cdot (1 - \xi).$$

Write $V := \ker(Tp) \subset T\mathbb{P}(E)$ for the bundle of tangent vectors of $\mathbb{P}(E)$ tangent to the fibres of $p\colon \mathbb{P}(E) \to N$, so that we have an exact sequence

$$(6) \qquad\qquad 0 \longrightarrow V \longrightarrow T\mathbb{P}(E) \longrightarrow p^*TN \longrightarrow 0.$$

This gives $c(\mathbb{P}(E)) = c(V)p^*c(N)$.

Moreover, V is isomorphic to the tensor product $Q \otimes l^*$, cf. [1, p. 281]. Thus, tensoring the sequence (3) with l^* yields

$$0 \longrightarrow \mathbb{C} \longrightarrow p^*E \otimes l^* \longrightarrow V \longrightarrow 0;$$

here $\mathbb{C}$ denotes a trivial complex line bundle. With the formula for computing the total Chern class of the tensor product with a line bundle [1, (21.10)] we find

$$(7) \qquad c(V) = c(p^*E \otimes l^*) = \sum_{i=0}^{r} p^*c_i(E)(1+\xi)^{r-i}.$$

Putting all this together, we have

$$\widetilde{\imath}^*(c(\widetilde{M}) - f^*c(M)) = p^*c(N)\left[\sum_{i=0}^{r} p^*c_i(E)(1+\xi)^{r-i}(1-\xi) - p^*c(E)\right].$$

Set

$$\overline{y} = -p^*c(N)\cdot\frac{1}{\xi}\left[\sum_{i=0}^{r} p^*c_i(E)(1+\xi)^{r-i}(1-\xi) - p^*c(E)\right] \in H^*(\mathbb{P}(E)),$$

so that

$$\widetilde{\imath}^*(c(\widetilde{M}) - f^*c(M)) = -\overline{y}\xi.$$

Lemma 4 then implies that

$$(8) \qquad c(\widetilde{M}) - f^*c(M) = \widetilde{\imath}^!(\overline{y}) + \lambda$$

with some class $\lambda \in H^*(\widetilde{M})$ satisfying $\widetilde{\imath}^*(\lambda) = 0$.

Write $\widetilde{\imath}_c$ for the inclusion $\widetilde{M} - \mathbb{P}(E) \to \widetilde{M}$. By the proof of the excision lemma (applied to the inclusions $\widetilde{\imath}$ and $\widetilde{\imath}_c$) we have $\widetilde{\imath}_c^*\widetilde{\imath}^! = 0$. Hence, by applying $\widetilde{\imath}_c^*$ to equation (8) we obtain

$$\begin{aligned}
\widetilde{\imath}_c^*(\lambda) &= \widetilde{\imath}_c^*(c(\widetilde{M}) - f^*c(M)) \\
&= c(\widetilde{M} - \mathbb{P}(E)) - f^*i_c^*c(M) \\
&= c(\widetilde{M} - \mathbb{P}(E)) - f^*c(M-N) = 0,
\end{aligned}$$

since f sends $\widetilde{M} - \mathbb{P}(E)$ diffeomorphically (and pseudoholomorphically) onto $M-N$. Again by the excision lemma, we know that there is a class $\gamma \in H^*(\mathbb{P}(E))$ with $\widetilde{\imath}^!(\gamma) = \lambda$. Then $\widetilde{\imath}^*\widetilde{\imath}^!(\gamma) = \widetilde{\imath}^*(\lambda) = 0$, so Lemma 10 provides us with a class $\beta \in H^*(N)$ such that

$$\lambda = \widetilde{\imath}^!(\gamma) = f^*i^!(\beta).$$

Together with equation (8) this means that we have proved the formula in Theorem 9 up to an extra term $f^*i^!(\beta)$ on the right-hand side. We call this the *weak blow-up formula*.

We now regard N as a submanifold in $M_S := M \times S^2$ with its natural product symplectic structure. The normal bundle of N in M_S is $E \oplus \mathbb{C}$, with $\mathbb{C}$ denoting a trivial complex line bundle. Write $\widetilde{M}_S$ for the blow-up of M_S along N, so that we have the following blow-up diagram:

$$\begin{array}{ccc}
\mathbb{P}(E \oplus \mathbb{C}) & \xrightarrow{\widetilde{\imath}_S} & \widetilde{M}_S \\
\downarrow{\scriptstyle p_S} & & \downarrow{\scriptstyle f_S} \\
N & \xrightarrow{i_S} & M_S.
\end{array}$$

Let l_S be the canonical line bundle over $\mathbb{P}(E \oplus \mathbb{C})$ and set $\xi_S = -c_1(l_S)$. We have $c_i(E \oplus \mathbb{C}) = c_i(E)$, in particular $c_{r+1}(E \oplus \mathbb{C}) = 0$. So the weak blow-up formula for this set-up reads

$$\text{(9)} \quad c(\widetilde{M}_S) - f_S^* c(M_S) =$$
$$= -\widetilde{i}_S^! \left[p_S^* c(N) \cdot \frac{1}{\xi_S} \left(\sum_{i=0}^{r} p_S^* c_i(E)(1 + \xi_S)^{r+1-i}(1 - \xi_S) - p_S^* c(E) \right) \right]$$
$$+ f_S^* i_S^!(\beta)$$

with some class $\beta \in H^*(N)$.

Consider the commutative diagram

$$
\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\ \widetilde{s}\ } & \widetilde{M}_S \\
\Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f_S} \\
M & \xrightarrow{\ s\ } & M_S,
\end{array}
$$

where s and $\widetilde{s}$ are the natural inclusion maps. We now apply $\widetilde{s}^*$ to the individual summands in equation (9). In the following computations we write ν_Y^X for the normal bundle of a submanifold $Y \subset X$.

We begin with the term $f_S^* i_S^!(\beta)$. Notice that $i_S = s \circ i$, hence $i_S^! = s^! i^!$. Moreover, by the self-intersection formula, the composition $s^* s^!$ equals taking the cup product with $c_1(\nu_M^{M_S}) = 0$. Hence

$$\text{(10)} \qquad \widetilde{s}^* f_S^* i_S^!(\beta) = f^* s^* i_S^!(\beta) = f^* s^* s^! i^!(\beta) = 0.$$

Next we deal with the two terms on the left-hand side of (9). Here we need a lemma.

Lemma 11. *The first Chern class of the normal bundle $\nu_{\widetilde{M}}^{\widetilde{M}_S}$ of $\widetilde{M}$ in $\widetilde{M}_S$ equals $-\widetilde{i}^!(1)$, i.e. minus the Thom class of $\mathbb{P}(E)$ in $\widetilde{M}$.*

Proof. Let M' be a parallel copy of M in M_S, so that $f^{-1}(M')$ is a diffeomorphic copy of M' in $\widetilde{M}_S$. From the explicit construction of the blow-up one sees that there is a singular chain of (real) codimension 1 in $\widetilde{M}_S$ whose boundary consists of $\widetilde{M}$ and $\mathbb{P}(E \oplus \mathbb{C})$ with their natural orientations (given by the complex structure) and $f^{-1}(M')$ with the reversed orientation. This chain can be taken as a smooth manifold with corner along the transverse intersection

$$\widetilde{M} \cap \mathbb{P}(E \oplus \mathbb{C}) = \mathbb{P}(E).$$

(For the construction of this codimension 1 chain, it is enough to replace M by $E \equiv E \oplus \{0\} \subset E \oplus \mathbb{C}$ and M' by a parallel copy $E' \equiv E \oplus \{\varepsilon\} \subset E \oplus \mathbb{C}$. Then consider the blow-up of $E \oplus \mathbb{C}$ along the zero section N. It suffices to deal with the case where N is a point, where this chain can be seen quite explicitly. It is best to visualise the blow-up by cutting out a ball B^{2r+2} centred at zero and of radius smaller than ε, and then collapsing its boundary S^{2r+1} under the Hopf map. A strip $E \times [0, \varepsilon]$ with boundary $E - E'$ will intersect that B^{2r+2} in half a ball

of dimension $2r + 1$. The intersection of S^{2r+1} with E is a $(2r-1)$-dimensional sphere Σ. Collapsing Σ gives the blow-up $\widetilde{M}$ of M. The intersection of S^{2r+1} with $E \times [0, \varepsilon]$ will be a $2r$-disc with boundary Σ. The interior of that disc is met by each Hopf fibre of S^{2r+1} exactly once. That disc will collapse, therefore, to the exceptional divisor $\mathbb{P}(E \oplus \mathbb{C})$.)

It follows that
$$[\widetilde{M}] + [\mathbb{P}(E \oplus \mathbb{C})] - [f^{-1}(M')] = 0$$
in $H_m(\widetilde{M}_S)$, where $m = \dim M$. Apply $PD_{\widetilde{M}_S}$ to this equation, which gives
$$\tau_{\widetilde{M}} + \tau_{\mathbb{P}(E \oplus \mathbb{C})} - \tau_{f^{-1}(M')} = 0$$
in $H^2(\widetilde{M}_S)$, where it is understood that these are the Thom classes of the respective inclusions into $\widetilde{M}_S$.

If two submanifolds A and B of a manifold X intersect transversely, then the pull-back of the Thom class τ_B^X to A is the Thom class $\tau_{A \cap B}^A$, see [2, pp. 371/2], but beware the misprint in formula (3) on page 372. Since $\widetilde{M}$ and $\mathbb{P}(E \oplus \mathbb{C})$ intersect transversely in $\mathbb{P}(E)$, and $\widetilde{M}$ does not intersect $f^{-1}(M')$, the lemma follows by applying $\widetilde{s}^*$ to the preceding equation and observing that $\widetilde{s}^*(\tau_{\widetilde{M}}) = \widetilde{s}^* \widetilde{s}^!(1) = c_1(\nu_{\widetilde{M}}^{\widetilde{M}_S})$ by the self-intersection formula. $\qquad \square$

Hence we get

(11) $$\widetilde{s}^* c(\widetilde{M}_S) = c(\widetilde{M}) c(\nu_{\widetilde{M}}^{\widetilde{M}_S}) = c(\widetilde{M}) \cdot (1 - \widetilde{i}^!(1))$$

and

(12) $$\widetilde{s}^* f_S^* c(M_S) = f^* s^* c(M_S) = f^*(c(M) c(\nu_M^{M_S})) = f^* c(M).$$

Finally, we come to the first summand on the right-hand side of (9). Consider the following commutative diagram:

$$
\begin{array}{ccccccc}
N & \xleftarrow{\;p\;} & \mathbb{P}(E) & \underset{\widetilde{\pi}}{\overset{\widetilde{i}_0}{\rightleftarrows}} & D(l) & \xrightarrow{\;\widetilde{i}_1\;} & \widetilde{M} \\
\equiv \downarrow & & i_{\mathbb{P}} \downarrow & & s|_{D(l)} \downarrow & & s \downarrow \\
N & \xleftarrow{\;p_S\;} & \mathbb{P}(E \oplus \mathbb{C}) & \underset{\widetilde{\pi}_S}{\overset{\widetilde{i}_{S0}}{\rightleftarrows}} & D(l_S) & \xrightarrow{\;\widetilde{i}_{S1}\;} & \widetilde{M}_S.
\end{array}
$$

Here $i_{\mathbb{P}}$ denotes the natural inclusion of $\mathbb{P}(E)$ in $\mathbb{P}(E \oplus \mathbb{C})$, and, as before, l_S the canonical line bundle over $\mathbb{P}(E \oplus \mathbb{C})$.

We claim that $s^* \widetilde{i}_S^! = \widetilde{i}^! i_{\mathbb{P}}^*$. This follows by considering the two squares on the right separately. Indeed, the equality $s^* \widetilde{i}_{S1}^! = \widetilde{i}_1^! (s|_{D(l)})^*$ is proved exactly like Lemma 6. The equality $(s|_{D(l)})^* \widetilde{i}_{S0}^! = \widetilde{i}_0^! i_{\mathbb{P}}^*$ follows from the observation that the Thom class $\widetilde{i}_{S0}^!(1)$ pulls back under $(s|_{D(l)})^*$ to the Thom class $\widetilde{i}_0^!(1)$; this in turn is a consequence of $s|_{D(l)}$ being an isomorphism on fibres and the characterisation of the Thom class as generator of the fibre (rel boundary) cohomology. Hence
$$(s|_{D(l)})^* \widetilde{i}_{S0}^!(.) = (s|_{D(l)})^* (\widetilde{\pi}_S^*(.) \widetilde{i}_{S0}^!(1)) = \widetilde{\pi}^* i_{\mathbb{P}}^*(.) \widetilde{i}_0^!(1) = \widetilde{i}_0^! i_{\mathbb{P}}^*(.).$$

From the two said equalities, the claim is immediate.

Furthermore, we have $i^*_{\mathbb{P}}(\xi_S) = \xi$. Thus, using all this information when we apply $\widetilde{s}^*$ to the first summand on the right-hand side of equation (9), we obtain

$$(13)\; -\widetilde{s}^* \widetilde{i}^!_S \left[p^*_S c(N) \cdot \frac{1}{\xi_S} \left(\sum_{i=0}^{r} p^*_S c_i(E)(1+\xi_S)^{r+1-i}(1-\xi_S) - p^*_S c(E) \right) \right] =$$

$$= \; -\widetilde{i}^! i^*_{\mathbb{P}} \left[p^*_S c(N) \cdot \frac{1}{\xi_S} \left(\sum_{i=0}^{r} p^*_S c_i(E)(1+\xi_S)^{r+1-i}(1-\xi_S) - p^*_S c(E) \right) \right]$$

$$= \; -\widetilde{i}^! \left[p^* c(N) \cdot \frac{1}{\xi} \left(\sum_{i=0}^{r} p^* c_i(E)(1+\xi)^{r+1-i}(1-\xi) - p^* c(E) \right) \right].$$

This expression equals the right-hand side of the blow-up formula we are aiming to prove, plus an extra summand

$$-\widetilde{i}^! \left[p^* c(N) \sum_{i=0}^{r} p^* c_i(E)(1+\xi)^{r-i}(1-\xi) \right] =$$

$$= \; -\widetilde{i}^! \left[p^* c(N) c(V) c(l) \right] \qquad \text{(equation (7))}$$

$$= \; -\widetilde{i}^! \left[c(\mathbb{P}(E)) c(l) \right] \qquad \text{(sequence (6))}$$

$$= \; -\widetilde{i}^! \widetilde{i}^* c(\widetilde{M}) \qquad \text{(sequence (1))}$$

Thus, from the weak blow-up formula and equations (10), (11) and (12) the blow-up formula follows if we can show that

$$c(\widetilde{M}) \widetilde{i}^!(1) = \widetilde{i}^! \widetilde{i}^* c(\widetilde{M}).$$

But this is simply the projection formula.

This concludes the proof of Theorem 9.

Remarks. (1) This blow-up formula was used by the second author in [12] to solve the geography problem for symplectic 8-manifolds.

(2) Our proof of the blow-up formula carries over to give the corresponding formula for the Stiefel-Whitney classes of any real manifold M blown up along a submanifold of codimension r (*sic!*). Simply read c_i as the Stiefel-Whitney class $w_i \in H^i(.;\mathbb{Z}_2)$, perform all computations in cohomology with coefficients in $\mathbb{Z}_2$, and in the final part of the proof above replace S^2 by S^1.

8. Special cases

We now derive explicit expressions for some Chern classes of blow-ups in a few special cases. We write the formulae so as to allow easy comparison with the expressions in [4, pp. 608–611], where an *ad hoc* method is used to derive the corresponding results for blow-ups of complex manifolds.

(1) *The first Chern class of arbitrary blow-ups:* Since $\mathbb{P}(E)$ is of codimension 2 in $\widetilde{M}$, the homomorphism $\widetilde{i}^!$ increases the cohomological degree by 2. So for the computation of c_1 we need to identify the degree 0 term inside the square brackets in the blow-up formula. This implies that $c_1(\widetilde{M}) - f^* c_1(M)$ equals $-\widetilde{i}^!(1)$ times the coefficient of the linear term in ξ in

$$(1+\xi)^r(1-\xi) = 1 + (r-1)\xi + \dots,$$

that is, $-(r-1)\widetilde{i}^!(1)$. Notice that $\widetilde{i}^!(1)$ equals the Poincaré dual of the class of the exceptional divisor $\mathbb{P}(E)$ in $\widetilde{M}$, so we have

$$c_1(\widetilde{M}) = f^*c_1(M) - (r-1)PD_{\widetilde{M}}[\mathbb{P}(E)].$$

(2) *Blow-up at a point:* Here N is a point and E is trivial, so (with $\dim M = 2r$)

$$
\begin{aligned}
c(\widetilde{M}) - f^*c(M) &= -\widetilde{i}^! \left[\frac{1}{\xi}\big((1+\xi)^r(1-\xi) - 1\big) \right] \\
&= -\widetilde{i}^! \sum_{\nu=0}^{r-1} \left(\binom{r}{\nu+1} - \binom{r}{\nu} \right) \xi^\nu.
\end{aligned}
$$

Set $\eta = -\widetilde{i}^!(1) = PD_{\widetilde{M}}[\mathbb{P}(E)]$. Then, using the projection and the self-intersection formula, we find

$$\eta^2 = -\eta \cdot \widetilde{i}^!(1) = -\widetilde{i}^!\widetilde{i}^*(\eta) = \widetilde{i}^!\widetilde{i}^*\widetilde{i}^!(1) = -\widetilde{i}^!(\xi),$$

and inductively

$$\eta^{\nu+1} = -\widetilde{i}^!(\xi^\nu).$$

Hence

$$c(\widetilde{M}) - f^*c(M) = \sum_{\nu=1}^{r} \left(\binom{r}{\nu} - \binom{r}{\nu-1} \right) \eta^\nu.$$

This is consistent with the well-known fact, cf. [11, p. 235], that blowing up a point is the same as taking the connected sum with a copy of $\overline{\mathbb{C}P}^r$, i.e. $\mathbb{C}P^r$ with the opposite of its natural orientation. For instance, we can verify the formula for the Euler characteristic,

$$\chi(M\#\overline{\mathbb{C}P}^r) = \chi(M) + \chi(\overline{\mathbb{C}P}^r) - 2 = \chi(M) + (r-1),$$

as follows. Since ξ is the positive generator of $H^2(\mathbb{P}(E))$, we have $\eta^r = -\widetilde{i}^!(\xi^{r-1}) = -PD_{\widetilde{M}}\widetilde{i}_*(1)$, which is the negative generator of $H^{2r}(\widetilde{M})$. So the formula for the Euler characteristic follows from

$$c_r(\widetilde{M}) - f^*c_r(M) = (1-r)\eta^r.$$

(3) *The second Chern class of a symplectic 6-manifold blown up along a 2-dimensional symplectic submanifold:* In this case, the blow-up formula becomes

$$
\begin{aligned}
c(\widetilde{M}) - f^*c(M) &= -\widetilde{i}^! \left[p^*c(N) \cdot \frac{1}{\xi} \left(\sum_{i=0}^{2} p^*c_i(E)(1+\xi)^{2-i}(1-\xi) - p^*c(E) \right) \right] \\
&= -\widetilde{i}^! \left[p^*c(N) \cdot \big(1 - \xi - (\xi^2 + p^*c_1(E)\xi + p^*c_2(E))\big) \right] \\
&= -\widetilde{i}^! \big(p^*c(N) \cdot (1-\xi) \big)
\end{aligned}
$$

by the fundamental relation (4). Hence

$$
\begin{aligned}
c_2(\widetilde{M}) &= f^*c_2(M) - \widetilde{i}^!(p^*c_1(N) - \xi) \\
&= f^*c_2(M) - \widetilde{i}^!(p^*i^*c_1(M) - p^*c_1(E) - \xi) \\
&= f^*c_2(M) - \widetilde{i}^!\widetilde{i}^*f^*c_1(M) + \widetilde{i}^!c_1(Q) \quad \text{(equation (5))} \\
&= f^*c_2(M) - f^*c_1(M)\widetilde{i}^!(1) + f^*i^!(1) \quad \text{(projection formula, \textit{formule clef})} \\
&= f^*(c_2(M) + PD_M[N]) - f^*c_1(M) \cdot PD_{\widetilde{M}}[\mathbb{P}(E)].
\end{aligned}
$$

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